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Localizability of relativistic particles in fuzzy phase space†

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Abstract. The concept of fuzzy phase space Γ_s, recently introduced in non-relativistic quantum mechanics, is extended to the relativistic case. The $L^2(\Gamma_s)$ representation of the wave packet is used as a basis for discussing localizability in Γ_s for non-zero mass particles as well as for the photon.

1. Introduction

It has been recently shown (Prugovečki 1976a) that if probability theory is generalized so as to deal with sample points which are 'fuzzy' rather than 'sharp', a consistent theory of measurement on fuzzy phase space $\hat{\Gamma}$ can be formulated in non-relativistic quantum mechanics; moreover time-dependent scattering theory in $\hat{\Gamma}$ is in complete agreement with conventional scattering theory in the realm of validity of the latter, while at the same time transcending some of its limitations (Prugovečki 1976b).

Quantum mechanics on fuzzy configuration space has also been considered (Ali and Doebner 1976), but although it provides a more realistic description of the actual position measurement process, it was found not to possess fundamentally new features which are not shared by its standard counterpart. In contradistinction, quantum mechanics in fuzzy phase space displays very definite information-theoretical advantages on account of the fact that, in the spinless case, position together with momentum constitute an *informationally* complete set of observables (Prugovečki 1976c).

Operationally, the concept of fuzzy phase space is based on the observation that the accuracy calibration of instruments used in the simultaneous *determinative* (Prugovečki 1967, 1973) measurement of position and momentum provides for each point $(\boldsymbol{q}, \boldsymbol{p})$ in phase space Γ a confidence function $\chi_{\boldsymbol{q},\boldsymbol{p}}$. If it were not for the uncertainty principle, one could postulate the existence of an ideal limit, namely that of a *perfectly accurate* instrument, whose confidence function at each point in Γ would be a δ function centred at that point. For actual instruments, however, the accuracy calibrations comply with the uncertainty relations, and we can postulate only the existence of the limiting case of *optimally accurate* instruments whose confidence functions are Gaussians which have standard deviations obeying those relations. Thus, if the calibrations are invariant under translations and rotations of the laboratory inertial frame of reference, then the resulting fuzzy phase spaces for a single particle are (taking $\hbar = 1$):

$$\Gamma_s = \{(\boldsymbol{q}, \chi_{\boldsymbol{q}}^{(s)}) \times (\boldsymbol{p}, \chi_{\boldsymbol{p}}^{(s^{-1})}) | \boldsymbol{q}, \boldsymbol{p} \in \mathbb{R}^3\}, \tag{1.1}$$

$$\chi_{\boldsymbol{v}}^{(\sigma)}(\boldsymbol{u}) = (\pi\sigma^2)^{-3/2} \exp[-\sigma^{-2}(\boldsymbol{u} - \boldsymbol{v})^2]. \tag{1.2}$$

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We refer to σ as the *spread* of the confidence function $\chi_v^{(\sigma)}$. Thus, we see that we are dealing with an entire spectrum $0<s<\infty$ of fuzzy phase spaces Γ_s; the limiting cases $s=0$ and $s=\infty$ can be identified with configuration and momentum space, respectively (Prugovečki 1976b).

The probability measures on Γ_s associated (Prugovečki 1976a) with Γ_s measurements on a spinless non-relativistic particle of mass $m>0$ in the state ψ can be expressed in terms of the probability density

$$\rho_\psi^{(s)}(\boldsymbol{q},\boldsymbol{p})=\int_{\mathbb{R}^6} w_\psi(\boldsymbol{x},\boldsymbol{k})\chi_{\boldsymbol{q}}^{(s)}(\boldsymbol{x})\chi_{\boldsymbol{p}}^{(s^{-1})}(\boldsymbol{k})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{k}, \tag{1.3}$$

where w_ψ is the Wigner transform of ψ:

$$w_\psi(\boldsymbol{x},\boldsymbol{k})=(2\pi)^{-3}\int_{\mathbb{R}^3}\psi^*(\boldsymbol{k}+\tfrac{1}{2}\boldsymbol{u})\exp(-\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{u})\psi(\boldsymbol{k}-\tfrac{1}{2}\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}. \tag{1.4}$$

In the relativistic case, however, the position observables are represented by the Newton–Wigner operators (Newton and Wigner 1949)

$$\boldsymbol{Q}=\mathrm{i}\nabla_k-\frac{1}{2}\frac{\mathrm{i}\boldsymbol{k}}{\boldsymbol{k}^2+m^2}. \tag{1.5}$$

Consequently, the marginality conditions which all probability densities in Γ_s have to satisfy (Prugovečki 1976a, § 5) impose on relativistic quantum mechanics a probability density $r_\psi^{(s)}$ that is different from $\rho_\psi^{(s)}$. This density is discussed in § 2.

Mass zero particles, such as the photon, have no non-relativistic counterpart. Hence in their case the concept of fuzzy phase space has to be re-examined in light of the limitations on the operational meaning that can be given to simultaneous measurement of their position and momentum. In § 3 we discuss and introduce the concept of localizability of the photon in fuzzy phase space Γ_s, and discuss its relationship to front localizability (Acharya and Sudarshan 1960) and to localizability in momentum space.

2. Localizability of non-zero mass particles in Γ_s

We shall treat in detail only fuzzy phase space for the Klein–Gordon particle since this case displays all the essential features of the general case. At the end of this section we indicate how this discussion extends to particles with non-zero spin.

As in the non-relativistic case, we adopt as basic the fuzzy phase spaces Γ_s in (1.1), with $0<s<\infty$. We note, however, that while in the context of non-relativistic physics the optimal confidence functions (1.2) were invariant under the fundamental group of space–time transformations, namely the Galilean group, that is not the case relativistically with respect to the corresponding Poincaré group. Indeed, (1.2) is left invariant by translations and rotations, but not by pure Lorentz transformations. Thus, the confidence functions in (1.2) can be considered to be outcomes of accuracy calibrations of instruments whose recording parts are at rest in the laboratory frame of reference, but they change (in an obvious manner) when those instruments are set in motion in relation to that frame.

To arrive at the counterpart of (1.3) for a Klein–Gordon particle, we rewrite (1.3) in the form

$$\rho_\psi^{(s)}(\boldsymbol{q},\boldsymbol{p};s)=|\psi(\boldsymbol{q},\boldsymbol{p};s)|^2, \tag{2.1}$$

by using the $L^2(\Gamma_s)$ representation (Prugovečki 1976b),

$$\psi(\boldsymbol{q}, \boldsymbol{p}; s) = (2\pi)^{-3/2}\langle\phi^{(s)}_{\boldsymbol{q},\boldsymbol{p}}|\psi\rangle, \tag{2.2}$$

$$\phi^{(s)}_{\boldsymbol{q},\boldsymbol{p}}(\boldsymbol{k}) = (\pi^{-1}s^2)^{3/4} \exp[-\tfrac{1}{2}s^2(\boldsymbol{k}-\boldsymbol{p})^2 - \mathrm{i}(\boldsymbol{k}-\tfrac{1}{2}\boldsymbol{p})\boldsymbol{q}], \tag{2.3}$$

of the state vector $\psi \in L^2(\mathbb{R}^3)$; the inner product in (2.2) corresponds to the momentum representation:

$$\langle\psi_1|\psi_2\rangle = \int_{\mathbb{R}^3} \psi_1^*(\boldsymbol{k})\psi_2(\boldsymbol{k})\,\mathrm{d}\boldsymbol{k}. \tag{2.4}$$

For a Klein–Gordon particle of rest mass m the momentum representation space is $L^2_\mu(\mathbb{R}^3)$, where

$$\mathrm{d}\mu\,(\boldsymbol{k}) = k_0^{-1}\,\mathrm{d}\boldsymbol{k}, \qquad k_0 = (\boldsymbol{k}^2 + m^2)^{1/2}, \tag{2.5}$$

and the inner product of two state vectors Ψ_1 and Ψ_2 is therefore

$$(\Psi_1|\Psi_2) = \int_{\mathbb{R}^3} \Psi_1^*(\boldsymbol{k})\Psi_2(\boldsymbol{k})\,\mathrm{d}\mu\,(\boldsymbol{k}). \tag{2.6}$$

Furthermore, in the relativistic case $\boldsymbol{Q}$ is given by (1.5), and $\boldsymbol{P}$ by

$$\boldsymbol{P}\Psi(\boldsymbol{k}) = \boldsymbol{k}\Psi(\boldsymbol{k}), \tag{2.7}$$

but these operators satisfy the same canonical commutation relation as their non-relativistic counterparts $\boldsymbol{Q}'$, $\boldsymbol{P}'$:

$$(\boldsymbol{Q}'\psi)(\boldsymbol{k}) = \mathrm{i}\nabla_{\boldsymbol{k}}\psi(\boldsymbol{k}), \qquad (\boldsymbol{P}'\psi)(\boldsymbol{k}) = \boldsymbol{k}\psi(\boldsymbol{k}). \tag{2.8}$$

Hence, by von Neumann's theorem (von Neumann 1931), there is a unitary operator U mapping $L^2(\mathbb{R}^3)$ into $L^2_\mu(\mathbb{R}^3)$ such that

$$\boldsymbol{Q} = U\boldsymbol{Q}'U^{-1}, \qquad \boldsymbol{P} = U\boldsymbol{P}'U^{-1}. \tag{2.9}$$

A simple computation shows that

$$(U\psi)(\boldsymbol{k}) = (\boldsymbol{k}^2 + m^2)^{1/4}\psi(\boldsymbol{k}). \tag{2.10}$$

Thus, the spectral density D for $(\boldsymbol{Q}, \boldsymbol{P})$ can be generated by means of U:

$$D(\boldsymbol{q}, \boldsymbol{p}; s) = UF(\boldsymbol{q}, \boldsymbol{p}; s)U^{-1}, \tag{2.11}$$

from the spectral density for $(\boldsymbol{Q}', \boldsymbol{P}')$,

$$F(\boldsymbol{q}, \boldsymbol{p}; s) = (2\pi)^{-3}|\phi^{(s)}_{\boldsymbol{q},\boldsymbol{p}}\rangle\langle\phi^{(s)}_{\boldsymbol{q},\boldsymbol{p}}|, \tag{2.12}$$

whose expectation value equals (2.1). Therefore, the probability density for finding a Klein–Gordon particle which is in the state Ψ at the fuzzy point $(\boldsymbol{q}, \chi^{(s)}_{\boldsymbol{q}}) \times (\boldsymbol{p}, \chi^{(s^{-1})}_{\boldsymbol{p}})$ is

$$r^{(s)}_\Psi(\boldsymbol{q}, \boldsymbol{p}) = (\Psi|D(\boldsymbol{q}, \boldsymbol{p}; s)\Psi) = (2\pi)^{-3}|\langle\phi^{(s)}_{\boldsymbol{q},\boldsymbol{p}}|U^{-1}\Psi\rangle|^2 \tag{2.13}$$

and in analogy with (2.12) we have

$$D(\boldsymbol{q}, \boldsymbol{p}; s) = (2\pi)^{-3/2}|U\phi^{(s)}_{\boldsymbol{q},\boldsymbol{p}})(U\phi^{(s)}_{\boldsymbol{q},\boldsymbol{p}}|. \tag{2.14}$$

We conclude that the $L^2(\Gamma_s)$ space for a Klein–Gordon particle consists of all the functions

$$\begin{aligned}\Psi(\boldsymbol{q},\boldsymbol{p};s)&=(2\pi)^{-3/2}(U\phi^{(s)}_{\boldsymbol{q},\boldsymbol{p}}|\Psi)\\&=(4\pi^3s^{-2})^{-3/4}\int\Psi(\boldsymbol{k})\exp[-\tfrac{1}{2}s^2(\boldsymbol{k}-\boldsymbol{p})^2+\mathrm{i}(\boldsymbol{k}-\tfrac{1}{2}\boldsymbol{p})\boldsymbol{q}]k_0^{-1/2}\,\mathrm{d}\boldsymbol{k},\end{aligned}\tag{2.15}$$

corresponding to all $\Psi(\boldsymbol{k})\in L^2_\mu(\mathbb{R}^3)$. As expected, the above expression is not explicitly covariant since neither (1.2) nor (1.5) are.

In view of (2.11), it follows right away that the expectation value (2.13) of D satisfies the marginality conditions

$$\int|\Psi(\boldsymbol{q},\boldsymbol{p};s)|^2\,\mathrm{d}\boldsymbol{q}=\int\chi^{(s^{-1})}_{\boldsymbol{p}}(\boldsymbol{k})|\Psi(\boldsymbol{k})|^2\,\mathrm{d}\mu(\boldsymbol{k})\tag{2.16}$$

on the fuzzy momentum space $\mathbb{R}^3_{(s^{-1})}=\{(\boldsymbol{p},\chi^{(s^{-1})}_{\boldsymbol{p}})|\boldsymbol{p}\in\mathbb{R}^3\}$, and

$$\int|\Psi(\boldsymbol{q},\boldsymbol{p};s)|^2\,\mathrm{d}\boldsymbol{p}=\int\chi^{(s)}_{\boldsymbol{q}}(\boldsymbol{x})|\hat{\Psi}(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}\tag{2.17}$$

$$\hat{\Psi}(\boldsymbol{x})=(2\pi)^{-3/2}\int\exp(\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{k})\Psi(\boldsymbol{k})k_0^{-1/2}\,\mathrm{d}\boldsymbol{k}\tag{2.18}$$

on the fuzzy configuration space $\mathbb{R}^3_s=\{(\boldsymbol{q},\chi^{(s)}_{\boldsymbol{q}})|\boldsymbol{q}\in\mathbb{R}^3\}$. This is so by virtue of the fact that F satisfies similar conditions (Prugovečki 1976a, b). In this context we note that $\hat{\Psi}(\boldsymbol{x})$ is indeed the configuration representation of Ψ, which corresponds to the position operators (1.5), i.e.,

$$(\Psi_1|\boldsymbol{Q}\Psi_2)=\int\boldsymbol{x}\hat{\Psi}_1^*(\boldsymbol{x})\hat{\Psi}_2(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.\tag{2.19}$$

This fact can be verified by means of equation (10) in the original paper by Newton and Wigner (1949). Furthermore, one can easily check by explicit computation that, just as in the non-relativistic case (cf equation (2.14) of Prugovečki 1976b), we have,

$$\lim_{s\to+\infty}(\pi s^2)^{3/4}\Psi(\boldsymbol{q},\boldsymbol{p};s)=k_0^{-1/2}\exp(-\tfrac{1}{2}\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{p})\Psi(\boldsymbol{p})\tag{2.20}$$

$$\lim_{s\to+0}(\pi s^{-2})^{3/4}\Psi(\boldsymbol{q},\boldsymbol{p};s)=\exp(\tfrac{1}{2}\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{p})\hat{\Psi}(\boldsymbol{q}).\tag{2.21}$$

Thus, apart from respective normalization factors, the probability densities in Γ_s become probability densities in momentum space and in configuration space in the limit $s\to+\infty$ and $s\to+0$, respectively.

The generalizations of the above results to the case of arbitrary spin are very straightforward if one uses the spinor approach advocated by Weidlich and Mitra (1963). When that is done, the inner product (2.6) generalizes to the inner product in their equation (1.30), and the role of the inverse U^{-1} of the operator (2.10) (effecting the transition to the corresponding non-relativistic case) is assumed by

$$\begin{aligned}&U^{-1}\{\phi^{(1)}(\boldsymbol{k})\times\ldots\times\phi^{(n)}(\boldsymbol{k})\}\\&\quad=m^{n/2}(\boldsymbol{k}^2+m^2)^{-(n+1)/2}\{(F_{\mathrm{W}}\phi^{(1)})(\boldsymbol{k})\times\ldots\times(F_{\mathrm{W}}\phi^{(n)})(\boldsymbol{k})\}\end{aligned}\tag{2.22}$$

where F_{W} denotes the Foldy–Wouthuysen transformation.

3. Fuzzy localizability of the photon

It had been observed already by Newton and Wigner (1949) that the problem of localizability of mass zero particles of spin 0 or $\frac{1}{2}$ could be dealt with in essentially the same manner as for $m \neq 0$ case, but that difficulties arose when the spin was greater than $\frac{1}{2}$. This fact gave rise to a whole series of papers discussing the spatial localizability of the photon (cf Amrein 1969 for a review), or attempting to provide a solution by discarding some of the restrictions imposed on position-coordinate observables, such as their commutability (Pryce 1948, Acharya and Sudarshan 1960) or the σ-additivity of their spectral measures (Jauch and Piron 1967). Therefore, in discussing the case of mass zero particles, we concentrate exclusively on the photon.

We start with the undisputed fact that the photon can be localized in momentum space. Working in the gauge

$$\boldsymbol{k}\boldsymbol{f}(\boldsymbol{k}) \equiv 0, \tag{3.1}$$

we can regard the Hilbert space $\mathscr{F}$ of all one-photon states as consisting of all vector-valued functions $\boldsymbol{f}(\boldsymbol{k})$ (with complex-valued components) which satisfy (3.1) and have a finite norm with respect to the inner product

$$(\boldsymbol{f}|\boldsymbol{g}) = \int_{\mathbf{R}^3} \boldsymbol{f}^*(\boldsymbol{k})\boldsymbol{g}(\boldsymbol{k})k_0^{-1}\,\mathrm{d}\boldsymbol{k}, \qquad k_0 = |\boldsymbol{k}|. \tag{3.2}$$

We shall study first the 'front' localizability of the photon. This presupposes an absolutely sharp measurement of the direction of motion $\Omega = \boldsymbol{k}/k_0$ of the photon. Thus we introduce the functions

$$\boldsymbol{f}_\Omega(k) = \boldsymbol{f}(\boldsymbol{k}), \qquad -\infty < k < +\infty, \qquad k = k_0 = |\boldsymbol{k}|, \tag{3.3}$$

which at fixed Ω can be considered to be the components of $\boldsymbol{f}(\boldsymbol{k})$ in the Hilbert space $\mathscr{F}_\Omega$ with inner product

$$(\boldsymbol{f}|\boldsymbol{g})_\Omega = \int_{-\infty}^{+\infty} \boldsymbol{f}_\Omega^*(k)\boldsymbol{g}_\Omega(k)k_0\,\mathrm{d}\boldsymbol{k} \tag{3.4}$$

corresponding to the direct integral decomposition

$$\mathscr{F} = \int^{\oplus} \mathscr{F}_\Omega\,\mathrm{d}\Omega. \tag{3.5}$$

We note that since k in (3.3) varies over the entire real line, the integration in Ω is over a unit hemisphere (say over $0 \leqslant \theta < \pi$, $0 \leqslant \phi < \pi$ if Ω is described by the spherical coordinates θ and ϕ and $\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$).

Consider now the possibility that instead of a sharp determination of the (oriented) magnitude p of the momentum $\boldsymbol{p} = p\Omega$ at given Ω, we have a fuzzy determination, described by the fuzzy point $(p, \chi_{p,\Omega}^{(s^{-1})})$:

$$\chi_{p,\Omega}^{(s^{-1})}(k) = (\pi^{-1}s^2)^{1/2}\exp[-s^2(k-p)^2]. \tag{3.6}$$

It is completely consistent with the uncertainty relations to assume that a simultaneous fuzzy determination of the position coordinate x in the direction Ω can be carried out with an accuracy described by the confidence function $\chi_{q,\Omega}^{(s)}(x)$. Hence, given $\boldsymbol{f}_\Omega(k) \in \mathscr{F}_\Omega$, the probability density for measuring the fuzzy point

$$(q, \chi_{q,\Omega}^{(s)}) \times (p, \chi_{p,\Omega}^{(s^{-1})}) \tag{3.7}$$

is obtained by using the expression for this same probability density in the Hilbert space $L^2(\mathbb{R}^1)\oplus L^2(\mathbb{R}^1)\oplus L^2(\mathbb{R}^1)$, and then making the transition to $\mathscr{F}_\Omega$ by means of the unitary transformation

$$(U_0\psi)(k)=k_0^{-1/2}\psi(k). \tag{3.8}$$

Thus, if we apply the method of § 2 to the present (one-dimensional) case, we obtain

$$(2\pi)^{-1}|(U_0\phi_{q,p}^{(s)}|f)_\Omega|^2=(2\pi)^{-1}\left|\int_{-\infty}^{+\infty}\phi_{q,p}^{(s)*}(k)f_\Omega(k)k_0^{1/2}\,\mathrm{d}k\right|^2, \tag{3.9}$$

$$\phi_{q,p}^{(s)}(k)=(\pi^{-1}s^2)^{1/4}\exp[-\tfrac{1}{2}s^2(k-p)^2-\mathrm{i}(k-\tfrac{1}{2}p)q], \tag{3.10}$$

for that probability.

It is easy to see that

$$\lim_{s\to+\infty}(\pi s^2)^{1/2}|(U_0\phi_{q,p}^{(s)}|f)_\Omega|^2=p_0|f_\Omega(p)|^2 \tag{3.11}$$

is the correct probability density for detecting a photon moving in the given direction Ω and having momentum of magnitude p. In fact, the mean value of $\boldsymbol{p}$ in the entire momentum space, i.e., the expectation value of $\boldsymbol{P}$, can be expressed in terms of (3.11):

$$\int \mathrm{d}\Omega\int \boldsymbol{p}|f_\Omega(p)|^2p_0\,\mathrm{d}p=\int \boldsymbol{p}|\boldsymbol{f}(\boldsymbol{p})|^2p_0^{-1}\,\mathrm{d}\boldsymbol{p}=\langle\boldsymbol{P}\rangle. \tag{3.12}$$

On the other hand,

$$\lim_{s\to+0}(\pi s^{-2})^{1/2}|U_0\phi_{q,p}^{(s)}|f)_\Omega|^2=(2\pi)^{-1}\left|\int_{-\infty}^{+\infty}\exp(\mathrm{i}qk)f_\Omega(k)k_0^{1/2}\,\mathrm{d}k\right|^2. \tag{3.13}$$

A straightforward computation establishes the mean values over the entire configuration space $\mathbb{R}^3$ of $\boldsymbol{q}=q\Omega$ with respect to the probability density (3.13) to be

$$\begin{aligned}\tfrac{1}{2}\int \mathrm{d}\Omega\int\left[\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}k}f_\Omega\right)^*f_\Omega+f_\Omega^*\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}k}f_\Omega\right)\right]k_0\,\mathrm{d}\boldsymbol{k}\\ =\tfrac{1}{2}\mathrm{i}\int[\boldsymbol{f}^*(\mathrm{i}\Omega\,.\,\nabla_{\boldsymbol{k}})\boldsymbol{f}-\boldsymbol{f}(\mathrm{i}\Omega\,.\,\nabla_{\boldsymbol{k}})\boldsymbol{f}^*]k_0^{-1}\,\mathrm{d}\boldsymbol{k}\\ =(\boldsymbol{f}|\tfrac{1}{2}\mathrm{i}(\nabla_{\boldsymbol{k}}\,.\,\hat{\boldsymbol{P}}+\hat{\boldsymbol{P}}\,.\,\nabla_{\boldsymbol{k}})\boldsymbol{f}).\end{aligned} \tag{3.14}$$

This equals the expectation value of the 'front' position operator Q^l introduced in the equations (25) and (31) of Acharya and Sudarshan (1960).

We note that both $\hat{P}$ and Q^l leave each Hilbert space $\mathscr{F}_\Omega$ in the decomposition (3.5) invariant, inducing there the respective operators P_Ω and Q_Ω, which are canonically conjugate. It is easy to check that (3.9) is actually the probability density of measuring the fuzzy point (3.7) for the simultaneous values of Q_Ω and P_Ω, and that it satisfies the usual marginality conditions with respect to fuzzy measurements of both Q_Ω and P_Ω.

We turn now our attention to the more general case when there exists some fuzziness in the determination of the direction of motion Ω.

Let $\mathscr{H}$ denote the Hilbert space with inner product (3.2) which consists of all functions $\boldsymbol{f}(\boldsymbol{k})$ of finite norm with respect to that inner product. The Hilbert space $\mathscr{F}$ of the photon is a closed subspace of $\mathscr{H}$. Let $\mathbb{P}$ denote the orthogonal projector of $\mathscr{H}$ onto $\mathscr{F}$. We define for $\boldsymbol{g}\in\mathscr{H}$

$$(U\boldsymbol{g})(\boldsymbol{k})=k_0^{1/2}\boldsymbol{g}(\boldsymbol{k}) \tag{3.15}$$

and adopt

$$G(\boldsymbol{q},\boldsymbol{p};s)=(2\pi)^{-3}\mathbb{P}|U\phi^{(s)}_{\boldsymbol{q},\boldsymbol{p}})(U\phi^{(s)}_{\boldsymbol{q},\boldsymbol{p}}|\mathbb{P} \tag{3.16}$$

as being the spectral density in Γ_s for the photon, where the projector $|\cdots)(\cdots|$ has to be applied to each component of the photon wave packet in accordance to (3.2). Thus

$$r_f^{(s)}(\boldsymbol{q},\boldsymbol{p})=(2\pi)^{-3}|(U\phi^{(s)}_{\boldsymbol{q},\boldsymbol{p}}|\boldsymbol{f})|^2 \tag{3.17}$$

is taken to be the probability density of detecting a photon in the state $\boldsymbol{f}$ at the fuzzy point $(\boldsymbol{q},\chi_{\boldsymbol{q}}^{(s)})\times(\boldsymbol{p},\chi_{\boldsymbol{p}}^{(s^{-1})})\in\Gamma_s$.

In addition to the fact that (3.17) is in keeping with (2.13), there are two other points that speak in its favour.

The first one is that (3.17) has the correct marginal distribution on the fuzzy momentum space $\mathbb{R}^3_{s^{-1}}$:

$$\int r_f^{(s)}(\boldsymbol{q},\boldsymbol{p})\,\mathrm{d}\boldsymbol{q}=\int \chi_{\boldsymbol{p}}^{(s^{-1})}(\boldsymbol{k})|\boldsymbol{f}(\boldsymbol{k})|^2\,\mathrm{d}\boldsymbol{k}. \tag{3.18}$$

The second point is that for photons that move in a given direction Ω, and whose state therefore is

$$\boldsymbol{f}(\boldsymbol{k}')=\boldsymbol{f}(\boldsymbol{k})k_0^{-1}\delta(\Omega-\Omega'),\qquad \Omega'=\boldsymbol{k}'/k_0', \tag{3.19}$$

(3.17) provides a result which is consistent with the probability density (3.9) for front fuzzy localizability in position and momentum. As a matter of fact, if we introduce the variables

$$k_\parallel=\boldsymbol{k}\,.\,\Omega,\qquad \boldsymbol{k}_\perp=\boldsymbol{k}-\Omega(\boldsymbol{k}\,.\,\Omega) \tag{3.20}$$

we easily compute that for $\boldsymbol{f}$ in (3.19) we have

$$r_f^{(s)}(\boldsymbol{q},\boldsymbol{p})=[(\pi^{-1}s^2)\exp(-s^2\boldsymbol{p}_\perp^2)] \times(4\pi^3s^{-2})^{-1/2}\left|\int\exp[-\tfrac{1}{2}s^2(k-p_\parallel)^2+\mathrm{i}(k-p_\parallel)(\Omega\,.\,\boldsymbol{q})]f_\Omega(k)k^{1/2}\,\mathrm{d}k\right|^2. \tag{3.21}$$

Comparison with (3.9) shows that the only difference lies in the presence of the term in square brackets, which does not contain $\boldsymbol{f}$ and is exclusively $\boldsymbol{p}_\perp$-dependent—its source being the fuzziness in the determination of Ω by the imperfectly accurate measurement device.

The marginal distribution in $(\boldsymbol{q},\chi_{\boldsymbol{q}}^{(s)})$ of an arbitrary state $\boldsymbol{f}$ is,

$$\int r_f^{(s)}(\boldsymbol{q},\boldsymbol{p})\,\mathrm{d}\boldsymbol{p}=\int\chi_{\boldsymbol{q}}^{(s)}(\boldsymbol{x})|\hat{A}(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}, \tag{3.22}$$

$$\hat{A}(\boldsymbol{x})=(2\pi)^{-3/2}\int\exp(\mathrm{i}\boldsymbol{x}\,.\,\boldsymbol{k})\boldsymbol{f}(\boldsymbol{k})k_0^{-1/2}\,\mathrm{d}\boldsymbol{k}. \tag{3.23}$$

Thus, the adoption of (3.17) as a probability density in fuzzy phase space implies that (3.22) is the photon probability density in fuzzy configuration space $\mathbb{R}^3_s$.

We note that this probability density has a well defined limit as $s\to+0$, namely $|\hat{A}(\boldsymbol{x})|^2$. It is therefore tempting to adopt $\hat{A}(\boldsymbol{x})$ as the configuration space representation of the one-photon state $\boldsymbol{f}(\boldsymbol{k})$. However, the corresponding spectral measure

$$(E(\Delta)\hat{A})(\boldsymbol{x})=\chi_\Delta(\boldsymbol{x})\hat{A}(\boldsymbol{x}) \tag{3.24}$$

in $\mathscr{H}$ does not leave $\mathscr{F}$ invariant. Actually, it is seen from (3.16) that the spectral measure corresponding to sharp measurements would have to be $\mathbb{P}E(\Delta)\mathbb{P}$. But the operators $\mathbb{P}E(\Delta)\mathbb{P}$, although positive-definite, are not projection operators, i.e., they do not arise from sharply localized states in the sense of Newton and Wigner (1949).

This last fact can be reconciled with the adoption of (3.22) as a probability density on fuzzy configuration space $\mathbb{R}_s^3$ either by acknowledging only that the photon is localizable in a fuzzy sense, without inferring from this that it is localizable in a sharp sense, or by relaxing the requirement that systems of imprimitivity related to sharp localizability should be given in terms of projector-valued measures to the weaker demand of POV-measures. The first point of view is in keeping with that recently advocated by Ali and Emch (1974), except that our analysis leads to a different spectral measure on fuzzy configuration space, namely to the marginal value on $\mathbb{R}_s^3$

$$\mathbb{P}\int_\Delta \chi_q^{(s)}(\boldsymbol{x})\,\mathrm{d}E_{\boldsymbol{x}}\mathbb{P}, \qquad \Delta\subset\mathbb{R}^3, \tag{3.25}$$

of the spectral measure on Γ_s whose spectral density is (3.16). The second point of view leads to $T_\Delta=\mathbb{P}E(\Delta)\mathbb{P}$ (with $E(\Delta)$ defined by (3.24)) as being the POV-measure for sharply localized photons—a suggestion that has been put forward by Kraus (1971).

4. Conclusion

In summary, we can state that for non-zero mass particles the concept of fuzzy phase space Γ_s plays the same role in relativistic as in non-relativistic quantum mechanics, namely it leads to $L^2(\Gamma_s)$ representation spaces for $0<s<\infty$ which are 'sandwiched' in between the configuration and momentum representations, with these last two being a kind of degenerate extreme.

For mass zero particles, like the photon, there is no general consensus as to what the configuration space representation of the wave packet is, and the argument involving the limit $s\to +0$ cannot be used in the same manner. However, the limit $s\to +\infty$ does lead to the accepted momentum representation. Moreover, for photons moving in a given direction Ω, the proposed probability density in Γ_s is consistent with the photon front localizability proposed by Acharya and Sudarshan (1960).

The fact that a concept of sharp front localizability of the photon can be introduced without difficulty while that of sharp *ordinary* localizability is unfeasible, indicates that the notion of direction of motion is intrinsic to the very concept of what a photon is, namely that it is an object moving in a specific direction. Indeed, if that is the case, by the uncertainty principle perfectly sharp localizability in configuration space is impossible since it would imply a total lack of information on the direction of motion. On the other hand, fuzzy phase space with its non-sharp information on momentum (and therefore also on direction of motion) does allow a corresponding non-sharp knowledge of position, and seems to be the most natural vehicle in the study of the question of localizability of the photon.

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